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COMPUTING THE ORDER OF STRENGTH IN TEAM TOURNAMENTS WHERE EACH TEAM DOES NOT COMPETE AGAINST EVERY OTHER

Abstract

In bridge team tournaments there often is not enough time for each team to compete against every other. Nevertheless, in order to obtain a final ranking order we modify a procedure proposed by Zermelo (1929) for chess tournaments. This algorithm can easily be implemented on any computer. Previous tests have proved that the differences of the so-called International Match Points (IMPs) occuring during rank computation are normally distributed with constant variance, whereas their expectations depend on the corresponding pair of teams competing against each other. We propose an algorithm to estimate this expectation from the known differences of the IMPs, including those for teams which have not met during the tournament.

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1. Rank versus strength

Let us suppose that $N \geq 2$ teams T_i, $1 \leq i \leq N$, participate in a bridge tournament. In general, each team does not necessarily compete against every other team. In each match between two teams - consisting of a fixed numer of games - each team reaches a result measured in International Match points (IMPs) which is then transformed in a

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tournament result of a certain amount of Victory Points (VPs). Thus the paper starts from the known set of IMPs for those teams which have competed against each other. Nevertheless, we offer a procedure for finding the final ranking via the usual sum of VPs gained by each team. Because of the given situation we define the symmetric index set

$$G := \{(i,j) \in \mathbb{N}_{[1,N]} \times \mathbb{N}_{[1,N]},\, T_i \text{ competes against } T_j\}$$

of the "real" matches, i.e. those matches which really took place. Here $\mathbb{N}_{[a,b]} := \{n \in \mathbb{N}_0 := \mathbb{N} + \{0\} : a \le n \le b\}$ denotes the set of natural numbers between a and b with 0 included. On the other hand we call the symmetric index set

$$G^c := \{(i,j) \in \mathbb{N}_{[1,N]} \times \mathbb{N}_{[1,N]},\, T_i \text{ does not compete against } T_j\}$$

of complementary elements the set of "virtual" matches. At the end of the tournament, we have a "matrix"

$$\delta = (\delta_{ij})_{(i,j)\in G} \text{ with } \delta_{ij} \in \mathbb{Z} := \{\text{integer numbers}\}, \tag{1.1}$$

where $\delta_{ij} \in \mathbb{Z}$ is the difference of the IMPs between T_i and $T_j, (i,j) \in G$. Thus $\delta_{ji} = -\delta_{ij}$, and T_i is called winning [losing] against T_j iff $\delta_{ij} \ge 0$ [$\delta_{ji} \le 0$]. The letter $\delta =$ difference is chosen in order to remind us of this fact. From the Victory Point Table of the World Bridge Federation (in short: IMP→VP-Table) one obtains a function

$$\mathbb{Z} \ni n \mapsto g_B(n) \in \mathbb{N}_{[0,30]}$$

(depending on the number B of played games - called "boards" in bridge) with whose help the Victory Points (= VPs) can be defined by

$$v_{ij} := g_B(\delta_{ij}),\, (i,j) \in G. \tag{1.2}$$

As an example for $g_B(\cdot)$ we quote Table 1.

Putting $v_{ij} :=$ "$*$" for $(i,j) \in G^c$ and $v_{ii} :=$ "$-$" we obtain from (1.2) the VP-matrix

$$v = (v_{ij})_{1\le i,j\le N}.$$

Further we set

$$v_i := \sum_{j=1}^{N} v_{ij},\; 1 \le i \le N, \tag{1.3}$$

TABLE 1: IMP→VP-Table for $B = 8$ (x/y for n reads as $x \leq n \leq y$)

n	$\leq$ -51	-50/-46	-45/-42	-41/-38	-37/-34	-33/-30	-29/-27	-26/-24
$g_B(n)$	0	1	2	3	4	5	6	7
n	-23/-21	-20/-18	-17/-15	-14/-12	-11/-9	-8/-6	-5/-2	-1
$g_B(n)$	8	9	10	11	12	13	14	15
n	0/1	2/5	6/8	9/11	12/14	15/17	18/20	21/23
$g_B(n)$	15	16	17	18	19	20	21	22
n	24/26	27/29	30/33	34/37	38/41	42/45	46/50	$\geq$ 51
$g_B(n)$	23	24	25	25	25	25	25	25

where stars and hyphens are counted as zeros. Hence v_i denotes the total number of VPs gained by T_i.

Suppose that v is a VP-matrix where each team competed against every other (hence the matrix does not contain stars). Then everybody will agree with the ranking order obtained simply by ordering the v_i according to size. (This is always possible in the case $N = 2$. We therefore assume from now on that $N \geq 3$.) Nevertheless, even in the case where each team competes against every other team it is not always possible to infer from $(v_i)_{1 \leq i \leq N}$ a reasonable order of strength for the competing teams. For suppose that a team tournament with $N = 4$ teams results in

$$v = \begin{bmatrix} - & 20 & 19 & 25 \\ 10 & - & 16 & 25 \\ 11 & 14 & - & 25 \\ 0 & 0 & 0 & - \end{bmatrix}$$

with

$$v_4 = 0 < v_3 = 50 < v_2 = 51 < v_1 = 64,$$

hence

$$T_4 < T_3 < T_2 < T_1. \tag{1.4}$$

However, since $v_4 = 0$, it makes no sense to derive an order of (relative) strengths from the ranking order (1.4). Even worse: assume that T_1 is the winner of the last World Championship, whereas the remaining teams are equally weak. Then the tournament could possibly result in

$$v = \begin{bmatrix} - & 25 & 25 & 25 \\ 0 & - & 15 & 15 \\ 0 & 15 & - & 15 \\ 0 & 15 & 15 & - \end{bmatrix}$$

yielding

$$v_4 = v_3 = v_2 = 30 < v_1 = 75$$

with $\min_{1 \le i \le 4} v_i \ne 0$. Hence the relative strength of T_1 compared to the other teams is $75/30 = 2.50$, where the relative strength of T_i against T_j is v_i/v_j. In the above eaxample we have $v_N = \cdots = v_2 = (N-2) \cdot 15 < v_1 = (N-1) \cdot 25$ with $N = 4$. Hence if the number of teams would tend to infinity, the relative strength v_1/v_N would tend to $25/15 = 1.67$ yielding the rather inappropriate result that the top team is less than twice as strong as the remaining weak teams. Admittedly, one could object that in a tournament "Each against every other" there is no need for computing the relative strengths. However, it is necessary to introduce the concept of relative strength as soon as each team does not compete against every other, for then it is definitely an advantage to play against a weak rather than a strong team, which could easily happen since opponents are selected at random. Moreover, as we have seen, it is by no means always possible to derive an order of strength from the ranking order, while the other way around is always possible.

2. Comparability of the teams

Assume that in a tournament there are two separate groups such that none of the teams in the first group plays against a team of the second group. This must not be allowed to happen under any circumstances because then the teams of the two groups cannot be compared any longer, even if it were somehow possible to design a mathematical procedure for the order of strength. As an example, consider the following VP-matrix v which could be the result of a tournament with two equally strong top teams T_1 and T_2 and two almost equally weak teams T_3 and T_4, namely

$$v = \begin{bmatrix} - & 15 & * & * \\ 15 & - & * & * \\ * & * & - & 16 \\ * & * & 14 & - \end{bmatrix}.$$

Recall that the stars indicate that the two top teams did not play against the two weak teams. Probably any procedure computing the order of strength would yield the absurd result that the weak team T_3 is the winner.

Comparability of the two teams can only be guaranteed by an appropriate matching strategy. Hence the set of rules (called "movement" in bridge) according to which the teams move from one round to the next have to be designed properly. In particular comparability is fullfilled if the following condition holds

$$\delta \text{ from (1.1) has non-zero entries in the two secondary diagonals.} \tag{2.1}$$

In [1] we supply together with the Zermelo algorithm, the "movement" for which (2.1) holds. This is guaranteed because the "movement" implies that after two rounds at the latest T_i has played against T_{i-1} and T_{i+1}.

3. IMPs versus VPs

In the sequel we set $\mathbb{R}$, $\mathbb{R}_{>a}$ and $\mathbb{R}_{\geq a}$ for the set of reals, namely the reals $> a$ and the reals $\geq a$, respectively. By $\mathbb{R}_{[a,b]}$ we denote the reals within the interval $[a, b]$. The interval also can be open or half-open. In an analogous notation we replace $\mathbb{R}$ by the set $\mathbb{Q}$ of rationals. By

$$\mathbb{N}_{[0,n]} \ni y \mapsto Bin(n, p, y) := \binom{n}{y} p^y (1-p)^{n-y}, \text{ in short } Bin(n, p) \text{ on } \mathbb{N}_{[0,n]},$$

we denote the discrete density of the sum of n i.i.d. (= independent and identically distributed) $\{0, 1\}$-random variables with success probability $p \in (0, 1)$. The procedure Zermelo [4] published in 1929 was designed for chess tournaments with a not necessarily constant number of games between the competitors. He started from (3.2) below which we interpret in the following way. The observation was a fixed pair (k, z) with

$$k = (k_{ij})_{(i,j)\in G} \text{ symmetric, } k_{ij} \in \mathbb{N}, \tag{3.1}$$

and

$$z = (z_{ij})_{(i,j)\in G},\ z_{ij} \in \mathbb{N}_{[0,k_{ij}]}, z_{ij} + z_{ji} = k_{ij}.$$

There z_{ij} denoted the number of games player i won against player j out of k_{ij} games. The result of Zermelo's procedure was a sequence $s^* = (s_i^*)_{i=1}^N$, $s_i^* > 0$, of "strengths" for the chess players depending on the observed result (k, z) in the following way: assume we start with some sequence $s = (s_i)_{i=1}^N, s_i > 0$, of "strengths" for the players. Then, from the standpoint of Probability Theory we could interpret the values

$$p_{ij}(s) := \frac{s_i}{s_i + s_j},\ (i,j) \in G, s \in S_{(0,\infty)} := \mathbb{R}^N_{>0},$$

as the probability that player i beats player j. Suppose that $(X_{ijn})_{n=1}^{k_{ij}}$ is a sequence of i.i.d. random variables $X_{ijn} \sim Bin(1, p_{ij}(s))$ representing the outcomes 1=gain or 0=loss of the n-th game of T_i against T_j. Then $Y_{ij} := \sum_{n=1}^{k_{ij}} X_{ijn}$ is the random number of games player i won against player j. Assume further that the elements of the vector $(Y_{ij})_{(i,j)\in G}$ are independent. Then the multidimensional discrete density of $(Y_{ij}, (i,j) \in G, i < j)$ on $\times_{(i,j)\in G, i<j} \mathbb{N}_{[0,k_{ij}]}$ equals

$$L_{Bin}(s) := \prod_{(i,j)\in G, i<j} Bin(k_{ij}, p_{ij}(s), \cdot) \text{ with } s \in S_{(0,\infty)}. \tag{3.2}$$

(Because of $k_{ij} = k_{ji}$ it suffices to consider L_{Bin} instead of the discrete density of $(Y_{ij})_{(i,j)\in G}$.) Then the density (3.2) evaluated at the observed result (k, z) with $z \in \times_{(i,j)\in G, i<j} \mathbb{N}_{[0,k_{ij}]}$ is the probability that z occurs when the unknown sequence of strengths equals s. It was maximized with respect to s. The maximum point, i.e. the maximum-likelihood estimator was chosen as s^*, thus depending on (k, z).

In 1987 the second author applied this theorem by Zermelo to team tournaments in his bridgeclub, as not all of the teams could play against each other in the weekly events due to lack of time. There $z_{ij} := v_{ij}$ from (1.2) denoted the VPs which team T_i won against team T_j and $k_{ij} := v_{ij} + v_{ji}$ denoted the total number of VPs between the same teams. In order to compute s^* on a computer he wrote a Turbo-Pascal program.

In section 4 we formulate a variant of the theorem of Zermelo and give a self-contained proof in section 6. In section 5 we apply the theorem to team tournaments not using the customary VPs as above but rely directly on $\delta = (\delta_{ij})_{i,j\in G}$ from (1.1). Actually, there the elements z_{ij} represent probabilities which depend on δ_{ij} by the definition

$$z_{ij} := \Phi_{0,B\tau^2}(\delta_{ij}),\ (i,j) \in G,\ \tau^2 := 5.5^2, \tag{3.3}$$

where

$$\mathbb{R} \ni x \mapsto \Phi_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-t^2/2} dt$$

denotes the (cumulative) distribution function of the $N(\mu, \sigma^2)$-distribution.

As a motivation for (3.3), we suppose that for fixed $(i,j) \in G$ the random vector $(D_{ijb})_{b=1}^B$ with D_{ijb} equal to the difference of the IMPs between T_i and T_j in each of the B played games is i.i.d. with mean μ_{ij} and variance τ_{ij}^2 both of them not known for the time being. From statistical observations, see e.g. [2] and [3], we learn that τ_{ij}^2 is independent of (i,j) and equals approximately 5.5^2. Hence we derive from the Central Limit Theorem that for large B the following approximation holds for the total sum $\Delta_{ij} := \sum_{b=1}^B D_{ijb}$ of the differences of the IMPs

$$P\left(\frac{\Delta_{ij} - B\mu_{ij}}{\sqrt{B}\tau} < x\right) \approx \Phi_{0,1}(x),\ x \in \mathbb{R}.$$

An easy calculation yields

$$P(\text{team } T_i \text{ wins against team } T_j) := P(\Delta_{ij} > 0) \approx \Phi_{0,B\tau^2}(B\mu_{ij}). \tag{3.4}$$

As an estimate for $B\mu_{ij}$ we use the observed value δ_{ij}. Thus

$$P(\text{team } T_i \text{ wins against team } T_j) = P(\Delta_{ij} > 0) \approx \Phi_{0,B\tau^2}(\delta_{ij}) \tag{3.5}$$

yielding (3.3) as our derived observation. In particular (3.3) and (2.1) yield that

$$\text{the two secondary diagonals of } (z_{ij}, (i,j) \in G) \text{ do not contain zeros.} \tag{3.6}$$

From (3.5) we obtain the probability distribution of Δ_{ij}, namely

$$\Delta_{ij} \sim N(\delta_{ij}, B\tau^2). \tag{3.7}$$

4. A variant of a theorem by Zermelo

In this section we present a variant of [4] which is more restrictive as we replace the fairly general condition in [4, Nr. 4, S.454] by (3.6). On the other hand this variant is more general, as real scores, not only integer ones, are admissible. But note Remark 1 below.

Given a fixed pair (k, z) with

$$k = (k_{ij})_{1\le i,j\le N, i\neq j} \text{ symmetric}, \;\; k_{ij} \in \mathbb{R}_{\ge 0}, \tag{4.1}$$

$$z = (z_{ij})_{1\le i,j\le N, i\neq j},\ z_{ij} \in \mathbb{R}_{[0,k_{ij}]}, z_{ij}+z_{ji} = k_{ij}, z_{ij} > 0 \text{ on the two secondary diagonals.} \tag{4.2}$$

Set

$$\begin{aligned} L(s) \;&:=\; \prod_{1\le i<j\le N} \frac{\Gamma(k_{ij}+1)}{\Gamma(z_{ij}+1)\cdot\Gamma(z_{ji}+1)} \cdot \left(\frac{s_i}{s_i+s_j}\right)^{z_{ij}} \cdot \left(\frac{s_j}{s_i+s_j}\right)^{z_{ji}} \\ &=:\; l \prod_{1\le i<j\le N} \left(\frac{s_i}{s_i+s_j}\right)^{z_{ij}} \cdot \left(\frac{s_j}{s_i+s_j}\right)^{z_{ji}},\ s \in S_{(0,\infty)} := \mathbb{R}^N_{>0}, \end{aligned} \tag{4.3}$$

with

$$l := \prod_{1\le i<j\le N} \frac{\Gamma(k_{ij}+1)}{\Gamma(z_{ij}+1)\cdot\Gamma(z_{ji}+1)} > 0,$$

where the Gamma-function $\Gamma(\cdot)$ is given by

$$\mathbb{R}_{>0} \ni x \mapsto \Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt.$$

With

$$p_{ij}(s) := \frac{s_i}{s_i+s_j} \in (0,\infty),\ 1\le i,j\le N,\ i\neq j,\ s \in S_{(0,\infty)}, \tag{4.4}$$

we obtain

$$L(s) = l \prod_{1\le i<j\le N} (p_{ij}(s))^{z_{ij}} \cdot (p_{ji}(s))^{z_{ji}}.$$

Put

$$r_i := \sum_{1\le j\le N} z_{ij} > 0,\ 1\le i\le N.$$

We now assert the following

Theorem (A variant of a result by Zermelo) (Existence and approximation of a maximum point)

Given (k,z) from (4.1) and (4.2) the following holds:

(a) $S_{(0,\infty)} \ni s \mapsto L(s)$ has a maximum point $s^* = (s_i^*)_{i=1}^N$. Hence

$$0 < s_i^* < \infty,\ 1 \le i \le N.$$

(b) For each $s = s^*$ the following set of equations is valid:

$$r_i = \sum_{1 \le j \le N, j \ne i} k_{ij} \cdot p_{ij}(s) > 0,\ 1 \le i \le N. \tag{4.5}$$

(c) Any solution s of (4.5) is uniquely determined except for constant positive factors.

(d) Set

$$f_i(s) := \frac{r_i}{\sum_{1 \le j \le N, j \ne i} \frac{k_{ij}}{s_i + s_j}} \in (0,\infty),\ 1 \le i \le N,\ s \in S_{(0,\infty)}. \tag{4.6}$$

Then s is a fixed point of $f(\cdot) = (f_i(\cdot))_{i=1}^N$, i.e. $s = f(s)$, iff s is a solution of (4.5). In particular,

$$s^* \text{ is a fixed point of } f(\cdot). \tag{4.7}$$

(e) The Zermelo procedure: define recursively for all $n \in \mathbb{N}_0$ the sequence of vectors $s^n = (s_i^n)_{i=1}^N \in S_{(0,\infty)}$ by

$$\begin{cases} s^0 & := & \vec{1}, \\ s^{n+1} & := & f(s^n) \in (0,\infty). \end{cases} \tag{4.8}$$

Then the following statements hold:

(e1) There exists $\lim_{n\to\infty} s^n =: \overline{s} \in S_{(0,\infty)}$,

(e2) $\overline{s}$ is a fixed point of $f(\cdot)$,

(e3) $\overline{s}$ is a maximum point of $s \mapsto L(s)$.

Remark 1 (Relation to Zermelo's result)

The function L of (4.3) extends L_{Bin} to the case of non-negative reals k_{ij} and z_{ij}. Thus we get back L_{Bin} in the integer case together with the property that s^* is a maximum-likelihood estimator. This property cannot be proved any longer in the real case. But in case $z_{ij} \in \mathbb{Q}_{\geq 0}$, there still exists a motivation for L: find the smallest common denominator d of all the $z_{ij}{}'s$ involved. Hence $\overline{k}_{ij} := dk_{ij} \in \mathbb{N}_0$, $\overline{z}_{ij} := dz_{ij} \in \mathbb{N}_{[0,\overline{k}_{ij}]}$ and $\overline{z}_{ij} + \overline{z}_{ji} = \overline{k}_{ij}$. Hence if we add upper indices in order to indicate dependency on (k, z) we obtain

$$L^{(k,z)}(s) = l^{(k,z)} \cdot (l^{(\overline{k},\overline{z})})^{-1/d} \cdot (L_{Bin}^{(\overline{k},\overline{z})}(s))^{1/d}.$$

As neither multiplication with a positive constant nor composition of $s \mapsto L_{Bin}^{(\overline{k},\overline{z})}(s)$ with the strictly increasing function $x \mapsto x^{1/d}$ on $\mathbb{R}_{\geq 0}$ affects maximum points, both of the functions $s \mapsto L^{(k,z)}(s)$ and $s \mapsto L_{Bin}^{(\overline{k},\overline{z})}(s)$ have the same maximum points. In section 5 we will apply the theorem to $k_{ij} = 1$ and $z_{ij} \in \mathbb{R}_{[0,k_{ij}]}$ by computing a maximum point. Hence, since all numbers on a computer are treated as rationals due to finiteness of the number of available digits, the mentioned motivation holds. Indeed, already in the integer case, the ratio $k_{ij} : z_{ij}$ of all played games compared with the games actually won is rational. The same interpretation is therefore admissible when starting from rationals instead of integers.

Remark 2 (Numerical methods)

There are at least three possibilities for computing a maximum point s^* in the above Theorem:

(i) One tries to solve (4.5) algebraically. This however will become tedious for large N.

(ii) One applies the successive approximation from (4.8) which by (e3) of the Theorem yields a solution.

(iii) One solves (4.5) using methods from Dynamic Programming.

5. Application of the Zermelo procedure to team tournaments

We now apply the Zermelo procedure in order to provide a ranking order for a tournament where not all of the teams have competed against each other. This we do step by step, at the same time supplying an example.

Step 1: We start with the matrix $\delta = (\delta_{ij})_{1 \le i,j \le N}$. According to (1.1) we know the $\delta_{ij} \in \mathbb{Z}, (i,j) \in G$, where $\delta_{ji} = -\delta_{ij}$, as soon as the tournament has terminated. Moreover, we set $\delta_{ij} :=$ "$*$" for the virtual matches $(i,j) \in G^c$ and $\delta_{ii} :=$ "$-$". As an example we consider

Example 1 (N=8 teams, 4 rounds, B=8 games)

$\delta =$

team	1	2	3	4	5	6	7	8
1	–	0	∗	4	∗	−14	∗	−14
2	0	–	2	∗	6	∗	−16	∗
3	∗	−2	–	4	∗	8	∗	−2
4	−4	∗	−4	–	20	∗	10	∗
5	∗	−6	∗	−20	–	8	∗	12
6	14	∗	−8	∗	−8	–	10	∗
7	∗	16	∗	−10	∗	−10	–	12
8	14	∗	2	∗	−12	∗	−12	–

Step 2: According to (3.3) we obtain the $z_{ij} > 0, (i,j) \in G$. Hence condition (2.1) holds trivially. Since $\delta_{ij} = -\delta_{ji}$,

$$z_{ij} + z_{ji} = \Phi_{0,B\tau^2}(\delta_{ij}) + \Phi_{0,B\tau^2}(\delta_{ji}) = 1,\ (i,j) \in G,$$

and we have $k_{ij} = 1, (i,j) \in G$. Setting $k_{ij} := z_{ij} := 0$ for $(i,j) \in G^c$ we obtain a pair (k,z) with (4.1) and (4.2). The entries zero for the virtual matches are to be considered only as starting values for the Zermelo iteration procedure. Basically, the procedure is a "balancing" process taking into account that not all of the teams played against each other.

Step 3: The Zermelo procedure now computes for each (k,z) a sequence $\overline{s} = (\overline{s}_i)_{i=1}^N$ of strengths (depending on (k,z)) for the sequence $(T_i)_{i=1}^N$ of teams. This value is the limit $\overline{s}$ of (4.8) where the iteration process stops as soon as there are no further changes with respect to the precision 10^{-5}. From parts (a), (b), (c) and (e3) of the theorem we know that there exists some $c > 0$ with $\overline{s} = cs^*$. Hence by (4.4)

$$p_{ij}(s^*) = p_{ij}(\overline{s}) = \frac{\overline{s}_i}{\overline{s}_i + \overline{s}_j},\ (i,j) \in G^c.$$

This value used by the procedure can be interpreted as the probability that T_i wins against T_j. Recall that by (3.3) and (3.5) the $z_{ij}, (i,j) \in G$, have the same interpretation. This justifies applying the inverse of the $N(0, B\tau^2)-$distribution on the way back in order to estimate the IMP-difference

$$\overline{\delta}_{ij} := \Phi^{-1}_{0,B\tau^2}(p_{ij}(\overline{s})),\ (i,j) \in G^c,$$

for the virtual matches while the IMP-difference for $(i,j) \in G$ are estimated by the observed values $\overline{\delta}_{ij} := \delta_{ij}$. Now the whole IMP-matrix $\overline{\delta}$ is known.

Step 4: Finally we derive estimates for the VPs $\overline{v}_{ij}$ from the $\overline{\delta}_{ij}$. Of course, for the real matches we use the customary IMP→VP-Table and define

$$\overline{v}_{ij} := g_B(\overline{\delta}_{ij}) = g_B(\delta_{ij}),\ (i,j) \in G.$$

For the virtual matches we use

$$\overline{v}_{ij} := \sum_{n=-\infty}^{\infty} g_B(n) \cdot \kappa_{ij}(n),\ (i,j) \in G^c, \tag{5.1}$$

with

$$\kappa_{ij}(n) := \Phi_{\overline{\delta}_{ij},B\tau^2}(n + \frac{1}{2}) - \Phi_{\overline{\delta}_{ij},B\tau^2}(n - \frac{1}{2}),\ n \in \mathbb{Z}, \tag{5.2}$$

i.e. according to (3.7), one assigns to the points $n \in \mathbb{Z}$ the mass which the $N(\overline{\delta}_{ij}, B\tau^2)$-distribution puts on the interval $[n - \frac{1}{2}, n + \frac{1}{2}]$. Thus $\overline{v}_{ij}$ is the expectation of $g_B(\Delta_{ij})$ where the variable Δ_{ij} has the discrete density κ_{ij}. Actually the sum in (5.1) is a finite sum. For $B = 8$ for example, we have $g_8(n) = 0$ for $n \leq -51$ and $g_8(n) = 25$ for $n \geq 30$. Hence (5.2) and Table 1 yield

$$\begin{aligned} \textstyle\sum_{-\infty}^{\infty} g_8(n) \cdot \kappa_{ij}(n) &= \textstyle\sum_{n=-50}^{29} g_8(n) \cdot \kappa_{ij}(n) + 25 \sum_{n=30}^{\infty} \kappa_{ij}(n) \\ &= \textstyle\sum_{n=-50}^{29} g_8(n) \cdot \kappa_{ij}(n) + 25(1 - \Phi_{\overline{\delta}_{ij},B\tau^2}(29.5)). \end{aligned}$$

In our Example 1 we obtain

$v =$

team	1	2	3	4	5	6	7	8	v_i
1	–	15.0	12.7	16.0	13.5	11.0	12.9	11.0	92.1
2	15.0	–	16.0	13.2	17.0	14.0	10.0	15.1	100.3
3	17.2	14.0	–	16.0	15.8	17.0	15.1	14.0	109.1
4	14.0	16.7	14.0	–	21.0	15.8	18.0	16.9	116.4
5	16.4	13.0	14.1	9.0	–	17.0	14.3	19.0	102.8
6	19.0	15.9	13.0	14.1	13.0	–	18.0	16.0	109.0
7	17.0	20.0	14.8	12.0	15.6	12.0	–	19.0	110.4
8	19.0	14.8	16.0	13.0	11.0	13.9	11.0	–	98.7

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Step 5: The sequence

$$\overline{v}_i := \sum_{j=1}^{N} \overline{v}_{ij},\ 1 \leq i \leq N,$$

from (1.3) will be ordered according to size.

Hence in our Example 1 we obtain

rank	1	2	3	4	5	6	7	8
number of team	4	7	6	3	5	2	8	1
VPs	116.4	110.4	109.1	109.0	102.8	100.3	98.7	92.1

6. Proof of Zermelo's theorem

Proof. Basically, the proof follows the ideas of Zermelo.

(a1) We first show that there exists a maximum point s^*. Setting $0/0 := 0$ and $0^0 := 1$ the function $L(\cdot)$ is defined and continuous on $S_{[0,\infty)}$. Suppose s lies on the border of $S_{[0,\infty)}$. Hence $s_i = 0$ for some $1 \leq i \leq N$. Thus when $1 \leq i \leq N-1$

$$0 \leq L(s_1, \cdots, s_{i-1}, 0, s_{i+1}, \cdots, s_N) \overset{(4.3)}{\leq} \left(\frac{s_i}{s_i + s_{i+1}}\right)^{z_{i,i+1}} \overset{(4.2)}{=} 0. \qquad (6.1)$$

When $i = N$ the following holds

$$0 \leq L(s_1, \cdots, s_{N-1}, 0) \overset{(4.3)}{\leq} \left(\frac{s_N}{s_{N-1} + s_N}\right)^{z_{N,N-1}} \overset{(4.2)}{=} 0$$

which together with (6.1) yields

$$L(\cdot) = 0 \text{ on the border of } S_{[0,\infty)}. \tag{6.2}$$

As $L(\cdot)$ is homogeneous of dimension 0 in s, i.e. $L(\lambda s) = L(s), \lambda > 0$, we can replace $S_{[0,\infty)}$ by the compact set $S_{[0,1]}$. Then the assertion follows by the Bolzano-Weierstraß Theorem. In particular $s^* < \infty$.

(a2) We prove that $s^* > 0$. Assume on the contrary that $s_i^* = 0$ for some $1 \leq i \leq N$. This by (6.2) yields the contradiction $L(s^*) = 0 < L(\vec{1})$.

(b) From (b) we infer that s^* is an inner point of $S_{[0,\infty)}$. Thus setting the partial derivatives of $s \mapsto ln(L(s))$ at s^* equal to zero yields (4.5).

(c) Note that the right hand side ρ_n in the set of equations

$$\sum_{i=1}^{n} \sum_{j=n+1}^{N} k_{ij} \cdot p_{ij}(s) \overset{(4.4)(4.5)}{=} \sum_{i=1}^{n} r_i - \sum_{i=1}^{n} \sum_{j=i+1}^{n} k_{ij} =: \rho_n,\ 1 \leq n \leq N, \tag{6.3}$$

is independent of $s \in S_{(0,\infty)}$. Now suppose that there are two different solutions $s, t \in S_{(0,\infty)}$, i.e. $s \neq \lambda t$ for all $\lambda > 0$. Order the elements of s and t by size and divide each of the sequences by the sum of their elements. Thus we obtain a sequence $u > 0$ with $\sum u_i = 1$ and a sequence $v > 0$ with $\sum v_i = 1$. Moreover, u and v are different points, hence without loss of generality $u_i < v_i$ for some i. Let n be the greatest of such $i's$. Then $1 \leq n < N$, since otherwise we obtain the contradiction $1 = \sum u_i < \sum v_i = 1$. Hence

$$0 < u_i < v_i,\ 1 \leq i \leq n, \text{ and } 0 < v_j \leq u_j,\ n+1 \leq j \leq N,$$

which yields

$$p_{ij}(u) = \frac{u_i}{u_i + u_j} < \frac{v_i}{v_i + v_j} = p_{ij}(v),\ 1 \leq i \leq n < j \leq N. \tag{6.4}$$

Thus

$$\begin{aligned}\rho_n &\overset{(6.3)}{=} \sum_{i=1}^{n}\sum_{j=n+1}^{N} k_{ij}\cdot p_{ij}(u)\\ &= \sum_{i=1}^{n-1}\sum_{j=n+1}^{N} k_{ij}\cdot p_{ij}(u) + k_{n,n+1}\cdot p_{n,n+1}(u) + \sum_{j=n+2}^{N} k_{nj}\cdot p_{nj}(u).\end{aligned} \tag{6.5}$$

Replacing u by v in (6.5) and using (3.6) and (6.4) yields the contradiction $\rho_n < \rho_n$.

(d) This follows immediately.

(e1) From (4.6) we obtain

$$r_i f_{ij}(s) := r_i \frac{\partial f_i(s)}{\partial s_j} = \begin{cases} \frac{k_{ij}}{(s_i+s_j)^2}\cdot f_i^2(s) \geq 0, & 1 \leq i,j \leq N, j \neq i,\\ \sum_{1\leq n\leq N, n\neq i} \frac{k_{in}}{(s_i+s_n)^2}\cdot f_i^2(s) \geq 0, & 1 \leq i \leq N, j = i.\end{cases} \tag{6.6}$$

Hence

$$s \mapsto f_i(s) \uparrow,\ 1 \leq i \leq N, \tag{6.7}$$

with respect to the product ordering. Now, fix $n \in \mathbb{N}_0$ and set

$$0 < \lambda^n := \min_{1\leq j\leq N} \frac{s_j^n}{s_j^*} \leq \max_{1\leq i\leq N} \frac{s_j^n}{s_j^*} =: \mu^n. \tag{6.8}$$

Hence

$$\lambda^n s_i^* \leq s_i^n \leq \mu^n s_i^*,\ 1 \leq i \leq N, \tag{6.9}$$

which implies

$$\lambda^n s^* \leq s^n \leq \mu^n s^*. \tag{6.10}$$

As f from (4.6) is homogeneous of dimension 1, i.e.

$$f(\lambda s) = (f_i(\lambda s))_{i=1}^N = (\lambda\cdot f_i(s))_{i=1}^N = \lambda\cdot f(s),\ \lambda > 0, \tag{6.11}$$

we obtain

$$\begin{aligned}\lambda^n s_i^* &\overset{(4.7)}{=} \lambda^n\cdot f_i(s^*) \overset{(6.11)}{=} f_i(\lambda^n s^*) \overset{(6.10)(6.7)}{\leq} f_i(s^n) \overset{(4.8)}{=} s^{n+1}\\ &\overset{(6.10)(6.7)}{\leq} f_i(\mu^n s^*) \overset{(6.11)}{=} \mu^n\cdot f_i(s^*) \overset{(4.7)}{=} \mu^n s_i^*,\ 1 \leq i \leq N.\end{aligned}$$

Hence

$$\lambda^n \le \frac{s^{n+1}}{s_i^*} \le \mu^n, 1 \le i \le N,$$

which, on taking the minimum and maximum with respect to i, yields

$$\lambda^0 \le \lambda^n \le \lambda^{n+1} \le \mu^{n+1} \le \mu^n \le \mu^0. \tag{6.12}$$

Hence

$$\overline{\lambda} := \lim_{n\to\infty} \lambda^n \text{ and } \overline{\mu} := \lim_{n\to\infty} \mu^n \le \mu$$

exist and

$$\lambda^n \le \overline{\lambda} \le \overline{\mu} \le \mu^n.$$

We claim that

$$\overline{\lambda} = \overline{\mu}. \tag{6.13}$$

Proof. (6.6) together with $f_i > 0$ from (4.6) and (4.2) imply that for each $1 \le i \le N$ there exists some $1 \le \rho_i \le N$ such that

$$f_{i\rho_i} > 0,\ 1 \le i \le N.$$

Hence continuity of all the functions involved guarantees that

$$f_{i\rho_i}|[\lambda^0 s^*, \mu^0 s^*] \ge \gamma_i > 0,\ 1 \le i \le N, \tag{6.14}$$

for some properly chosen γ_i. Hence we infer from the mean value theorem

$$\begin{aligned} s_i^{n+1} - \lambda^n s_i^* = f_i(s^n) - \lambda^n s_i^* &\overset{(4.7)(6.11)}{=} f_i(s^n) - f_i(\lambda^n s^*) \\ &=: \sum_{j=1}^{N} f_{ij}(x^i)\cdot(s_j^n - \lambda^n s_j^n),\ 1 \le i \le N, \end{aligned} \tag{6.15}$$

where x^i denotes an adequately chosen point on the straight line between $\lambda^n s^*$ and s^n which implies

$$x^i \in [\lambda^n s^*, s^n] \overset{(6.10)}{\subset} [\lambda^n s^*, \mu^n s^*] \overset{(6.12)}{\subset} [\lambda^0 s^*, \mu^0 s^*].$$

Hence

$$s_i^{n+1} - \lambda^n s_i^* \overset{(6.15)}{\ge} f_{i\rho_i}\cdot(s_{\rho_i}^n - \lambda^n s_{\rho_i}^*) \overset{(6.14)}{\ge} \gamma_i\cdot(s_{\rho_i}^n - \lambda^n s_{\rho_i}^*),\ 1 \le i \le N. \tag{6.16}$$

Setting

$$\gamma := \min_{1 \le i \le N} \frac{\gamma_i s^*_{\rho_i}}{s^*_i} > 0$$

we obtain

$$\lambda^{n+1} - \lambda^n \overset{(6.8)}{\ge} \frac{s_i^{n+1}}{s_i^*} - \lambda^n \overset{(6.16)}{\ge} \frac{\gamma_i s^*_{\rho_i}}{s^*_i} \cdot \left(\frac{s^n_{\rho_i}}{s^*_{\rho_i}} - \lambda^n \right) \ge \gamma \left(\frac{s^n_{\rho_i}}{s^*_{\rho_i}} - \lambda^n \right)$$

which implies

$$\lambda^{n+1} - \lambda^n \overset{(6.8)}{\ge} \gamma(\mu^n - \lambda^n).$$

For $n \to \infty$ we obtain (6.13). Thus (6.9) yields

$$\lim_{n\to\infty} s_i^n = \overline{\lambda} s_i^*,\ 1 \le i \le N.$$

(e2) This assertion follows from (4.8) by (e1) and continuity of (4.6).

(e3) By (e2) $\overline{s}$ is a fixed point of $f(\cdot)$. Hence by (d) $\overline{s}$ is also a solution of (4.5). By (c) this solution is unique except for constant positive factors. Hence $\overline{s} = cs^*$ for some $c > 0$. Thus also $\overline{s}$ is a maximum point of $s \mapsto L(\cdot)$ as by (a) s^* is a maximum point and $L(\cdot)$ is homogeneous of order 0.

7. Software

The C++ code as well as a DOS version of the complete algorithm for the Zermelo procedure may be downloaded cost-free from the internet address given in [1]. Included are all IMP→VP-Tables and an observation-file as an example.

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